UNIT-03

21/08/2025 **GAMMA AND BETA DISTRIBUTIONS** Thursday

***Gamma distribution:-** to model the waiting time until the occurence of a specified no. of events in a poisson process.

A continuous random variable x in the limits $(0,\infty)$ has the probability density function is given by $f(x)$. [Γ — Small gamma]

$$f(x) = \begin{cases} \frac{1}{\Gamma(\lambda)} e^{-x} \cdot x^{\lambda-1} & ; x \geq 0 \\ & ; \lambda \geq 0 \\ 0 & ; \text{Otherwise} \end{cases}$$

is called gamma distribution with parameter λ. It is usually denoted by $x \sim G(\lambda)$.

***Gamma Integrals:-** From the distribution function we have $\int_0^\infty f(x)\,dx = 1$

$$\int_0^\infty \frac{1}{\Gamma(\lambda)} e^{-x} \cdot x^{\lambda-1}\,dx = 1$$

If a machine fails on average once every 2 months and we want the distribution of time until it fail 3 times

$$\frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-x} x^{\lambda-1}\,dx = 1$$

$$\int_0^\infty e^{-x} x^{\lambda-1}\,dx = \Gamma(\lambda)$$

//

Moments :

We know that the r^{th} moment at the origin is denoted by μ_r' and is defined as

$$\mu_r' = E(x^r)$$

$$= \int_0^\infty x^r f(x)\, dx$$

$$= \int_0^\infty x^r \frac{1}{\Gamma(\lambda)} e^{-x} x^{\lambda-1} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\lambda)} e^{-x} x^{(r+\lambda)-1} dx \quad \text{(from Gamma integrals)}$$

$$= \int_0^\infty \frac{1}{\Gamma(\lambda)} e^{-x} x^{(r+\lambda)-1} dx. \qquad \int_0^\infty e^{-x} x^{n-1} dx = \Gamma n$$

$$\boxed{\mu_r' = \frac{1}{\Gamma(\lambda)} \Gamma(\lambda+r)}$$

if $r=1$

$$\mu_1' = \frac{1}{\Gamma(\lambda)} \Gamma(\lambda+1) \qquad \left[\because \Gamma(\lambda+1) = \frac{\Gamma(\lambda)\, \lambda!}{(\lambda-1)!}\right]$$

$$= \frac{1}{\Gamma(\lambda)} \Gamma\lambda \frac{\lambda!}{(\lambda-1)!}$$

$$\mu_1' = \frac{\lambda!}{(\lambda-1)!} = \frac{\lambda(\lambda-1)!}{(\lambda-1)!}$$

$$\boxed{\mu_1' = \lambda} = \text{Mean}$$

if $r = 2$

$$\mu_2' = \frac{1}{\Gamma(\lambda)} \Gamma(\lambda+2)$$

$$= \frac{1}{\cancel{\Gamma(\lambda)}} \cancel{\Gamma(\lambda)} \frac{(\lambda+1)!}{(\lambda-1)!}$$

$$= \frac{(\lambda+1)\lambda\,(\lambda-1)!}{(\lambda-1)!}$$

$$\boxed{\mu_2' = \lambda\cdot(\lambda+1)}$$

if $r=3$

$$\mu_3' = \frac{1}{\Gamma(\lambda)} \Gamma(\lambda+3)$$

$$= \frac{1}{\Gamma(\lambda)} \frac{\Gamma(\lambda)(\lambda+2)}{(\lambda-1)!}$$

$$= \frac{(\lambda+2)(\lambda+1)(\lambda)(\lambda-1)!}{(\lambda-1)!}$$

$$\boxed{\mu_3' = \lambda(\lambda+1)(\lambda+2)}$$

if $r = 4$

$$\mu_4' = \frac{1}{\Gamma(\lambda)} \Gamma(\lambda+4)$$

$$= \frac{1}{\Gamma(\lambda)} \Gamma(\lambda) \frac{\lambda+3!}{(\lambda-1)!}$$

$$= \frac{(\lambda+3)(\lambda+2)(\lambda+1)(\lambda)\ (\lambda-1)!}{(\lambda-1)!}$$

$$\boxed{\mu_4' = \lambda(\lambda+1)(\lambda+2)(\lambda+3)}$$ //

* Central moments :-

$\mu_1 = 0$

Imp

$\mu_2 = \mu_2' - (\mu_1')^2$

$= \lambda(\lambda+1) - \lambda^2$

$= \lambda^2 + \lambda - \lambda^2$

$\boxed{\mu_2 = \lambda}$ // variance.

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= \lambda(\lambda+1)(\lambda+2) - 3\lambda(\lambda+1)\lambda + 2\lambda^3$

$= \lambda[\lambda^2+\lambda+2\lambda+2] - 3(\lambda^3+\lambda^2) + 2\lambda^3$

$= \lambda^3 + \lambda^2 + 2\lambda^2 + 2\lambda - 3\lambda^3 - 3\lambda^2 + 2\lambda^3$

$\mu_3 = \cancel{-2\lambda^3}\ 2\lambda$

$\boxed{\mu_3 = 2\lambda}$ //

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= \lambda(\lambda+1)(\lambda+2)(\lambda+3) - 4\lambda(\lambda+1)(\lambda+2)\lambda + 6[\lambda(\lambda+1)(\lambda^2)]$

$= \lambda[(\lambda^2+\lambda+2\lambda+2)(\lambda+3)] - 4\lambda^2(\lambda+1)(\lambda+2) + 6\lambda^3(\lambda+1) - 3\lambda^4$

$= \lambda[\lambda^3+\lambda^2+2\lambda^2+2\lambda+3\lambda^2+3\lambda+6\lambda+6]$

$= 4\lambda^3(\lambda^2+\lambda+2\lambda+2) + 6\lambda^4+6\lambda^3-3\lambda^4$

$= \lambda^4+\lambda^3+2\lambda^3+2\lambda^2+3\lambda^3+3\lambda^2+6\lambda^2+6\lambda-4\lambda^4-4\lambda^3-8\lambda^3-8\lambda^2+6\lambda^4+6\lambda^3-3\lambda^4$

$= 6\lambda^3+3\lambda^2$

$= 3\lambda(\lambda+2)$

$$\boxed{\mu_4 = 3\lambda(\lambda+2)}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{4\lambda^2}{\lambda^3} = \frac{4}{\lambda}$$

$$\boxed{\beta_1 = \frac{4}{\lambda}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda(\lambda+2)}{\lambda^2} = \frac{3(\lambda+2)}{\lambda}$$

$\gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{4}{\lambda}} = \frac{2}{\sqrt{\lambda}} \Rightarrow$ Skewness

$$\gamma_2 = \beta_2 - 3 = \frac{3(\lambda+2)}{\lambda} - 3 = \frac{3\lambda+6-3\lambda}{\lambda} = \frac{6}{\lambda}$$

γ_2 and β_2 are considered as kurtosis

M.G.F of Gamma distribution :-

we know that M.G.F of Gamma distribution is denoted by $M_x(t)$ and is defined as

$$M_X(t) = E[e^{-tx}]$$

$$= \int_0^\infty e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \frac{1}{\Gamma(\lambda)} e^{-x}\,dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{tx} e^{-x} x^{\lambda-1}\,dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{x(t-1)} x^{\lambda-1}\,dx$$

$$M_X(t) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(1-t)x} x^{\lambda-1}\,dx \quad \text{--- (1)}$$

let $(1-t)x = y \Rightarrow x = \frac{y}{1-t}$

$$(1-t)\,dx = dy$$

$$dx = \frac{1}{1-t}\,dy$$

Sub above terms in eq (1)

$$M_X(t) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-y} \left(\frac{y}{1-t}\right)^{(\lambda-1)} \frac{dy}{(1-t)}$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-y} \frac{y^{\lambda-1}}{(1-t)^{\lambda}\cancel{(1-t)^{\lambda-1}}} \frac{dy}{\cancel{1-t}}$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-y} \, y^{\lambda-1} (1-t)^{-\lambda} \, dy$$

$$= \frac{(1-t)^{-\lambda}}{\Gamma(\lambda)} \int_0^\infty e^{-y} \, y^{\lambda-1} \, dy$$

$$= \frac{(1-t)^{-\lambda}}{\cancel{\Gamma(\lambda)}} [\cancel{\Gamma(\lambda)}] \quad [\because \text{ Gamma integral}]$$

$$\boxed{M_X(t) = (1-t)^{-\lambda}}$$

Imp

* characteristic function of gamma distribution:

we know the C.F of gamma distribution is denoted by $\phi_X(t)$, it is defined as

$$\phi_X(t) = E[e^{itx}]$$

$$= \int_0^\infty e^{itx} f(x) \, dx$$

$$= \int_0^\infty e^{itx} \frac{1}{\Gamma(\lambda)} e^{-x} \cdot x^{\lambda-1} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\lambda)} e^{(it-1)x} x^{\lambda-1} dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{x(it-1)} x^{\lambda-1} dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(1-it)x} x^{\lambda-1} dx \quad -(1)$$

let $(1-it)x = y \Rightarrow x = \frac{y}{1-it}$

$(1-it)dx = dy \Rightarrow dx = \frac{1}{1-it} dy$

Sub above terms in eq ①

$$\phi_x(t) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-y} \left(\frac{y}{1-it}\right)^{\lambda-1} \frac{dy}{1-it}$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-y} \frac{y^{\lambda-1}}{(1-it)^{\lambda-1}} \frac{dy}{1-it}$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-y} \frac{y^{\lambda-1}}{(1-it)^{\lambda}} dy$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-y} y^{\lambda-1} (1-it)^{-\lambda} dy$$

$$\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n$$

$$= \frac{(1-it)^{-\lambda}}{\Gamma(\lambda)} \int_0^\infty e^{-y} y^{\lambda-1} dy \quad [\because \text{Gamma integrals}]$$

$$\phi_x(t) = \frac{(1-it)^{-\lambda}}{\Gamma(\lambda)} [\Gamma(\lambda)]$$

$$\boxed{\phi_x(t) = (1-it)^{-\lambda}}$$

4

* C.G.F of Gamma distribution:-

We know that CGF is denoted by

$K_x(t)$ ~~is it~~ is defined as

$$K_x(t) = \log[M_x(t)]$$

$$= \log(1-t)^{-\lambda}$$

$$= \log(1-t)^{-1\lambda}$$

$$= \lambda \log(1-t)^{-1}$$

We know that

$$\log(1-x)^{-1} = \left[x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$

$$K_x(t) = \lambda\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right]$$

$$K_1\frac{t}{1} + K_2\frac{t^2}{2!} + K_3\frac{t^3}{3!} + \cdots =$$

$$\lambda t + \frac{\lambda t^2}{2!} + \frac{\lambda t^3}{3!} + \frac{\lambda t^4}{4!} + \cdots$$

Comparing the coefficients

$$K_1\frac{t^1}{1} = \lambda t \qquad K_2\frac{t^2}{\cancel{2!}} = \lambda\frac{t^2}{\cancel{2!}}$$

$\boxed{K_1 = \lambda} =$ Mean, $\boxed{K_2 = \lambda}$. Variance

$$K_3\frac{t^3}{3!} = \lambda\frac{t^3}{3}$$

$$K_3\frac{t^3}{6} = \frac{\lambda t^3}{3} \Rightarrow \boxed{\mu_3 = K_3 = 2\lambda}$$

$$K_4 \frac{t^4}{4!} = \frac{\lambda t^4}{4}$$

$$K_4 \frac{t^4}{24} = \frac{\lambda t^4}{4}$$

$$K_4 = 6\lambda$$

$$K_4 = \mu_4 - 3K_2^2$$

$$\mu_4 = K_4 + 3K_2^2$$

$$= 6\lambda + 3\lambda^2$$

$$\boxed{\mu_4 = 3\lambda(\lambda+2)}$$

* Reproductive & additive property of Gamma distribution :-

Statement :- The sum of k independent gamma variates is also a gamma variate.

Let $X_1, X_2 \ldots X_K$ are k independent with parameters are $\lambda_1, \lambda_2, \ldots \lambda_K$.

We know that

M.G.F of Gamma distribution

$$M_X(t) = (1-t)^{-\lambda}$$

Now M.G.F for $X_1, X_2, \ldots X_K$ is

$$M_X(t) = (1-t)^{-\lambda}$$

$$M_{X_1}(t) = (1-t)^{-\lambda_1}$$

$$M_{X_2}(t) = (1-t)^{-\lambda_2}$$

$$\vdots$$

$$M_{X_K}(t) = (1-t)^{-\lambda_K}$$

let us consider

$$M_{X_1+X_2+\cdots X_K}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_K}(t)$$

$$M_{\sum_{i=1}^{K} X_i}(t) = (1-t)^{-\lambda_1} \cdot (1-t)^{-\lambda_2} \cdots (1-t)^{-\lambda_K}$$

$$= (1-t)^{-\lambda_1-\lambda_2 \cdots -\lambda_K}$$

$$= 1-t^{-[\lambda_1+\lambda_2+\cdots \lambda_K]}$$

$$M_{\sum_{i=1}^{K} X_i}(t) = (1-t)^{-\sum_{i=1}^{K} \lambda_i}$$

this is the M.G.F of Gamma distribution.

Hence by uniqueness theorem of the distributio $x_1+x_2 \cdots x_K$ will is also a gamma variate with parameters $\lambda_1+\lambda_2 \cdots \lambda_K$.

22/08/2025

* Beta distribution :-

* Beta distribution of first kind:-

let x be continuous random Variable in the interval $(0, 1)$ has the probability density function

$$F(x) = \begin{cases} \frac{1}{\beta(m,n)} x^{m-1} (1-x)^{n-1} & ; 0 < x < 1, m, n \ge \\ 0 & ; \text{otherwise} \end{cases}$$

is called Beta distribution of first kind with parameters m and n. It is usually denoted by $x \sim \beta(m, n)$

*Beta distribution of first kind integrals:-

$$\int_0^1 f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 \frac{1}{\beta(m,n)} x^{m-1} (1-x)^{n-1}\,dx = 1$$

$$\Rightarrow \int_0^1 x^{m-1} (1-x)^{n-1}\,dx = \beta(m,n)$$

* Beta distribution of Second kind:-

let x be a continuous random variable

with $(0,\infty)$ and has probability density function

$$f(x) = \begin{cases} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}}; & 0 < x < \infty \\ 0 & ; \text{Otherwise} \end{cases}$$

is called Beta distribution of second kind with parameters m and n. It is denoted by $x \sim \beta_2(m,n)$.

* Beta distribution of Second kind integrals:

we know that

$$\int_{0}^{\infty} f(x)\, dx = 1$$

$$\int_0^{\infty} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} dx = 1$$

$$\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \beta(m,n)$$

* Harmonic mean of Beta distribution of second kind:-

Harmonic mean H is calculated by

$$\frac{1}{H} = \int \frac{1}{x} f(x)\, dx$$

$$= \int_0^{\infty} \frac{1}{x} \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \frac{1}{\beta(m,n)} \int_0^\infty \frac{x^{m-1+1}}{(1+x)^{m+n}} dx$$

$$= \frac{1}{\beta(m,n)} \int_0^\infty \frac{x^{m-1}}{(1+x)^{m-1+n+1}} dx$$

$$= \frac{1}{\beta(m,n)} \cdot \beta(m-1, n+1)$$

$$= \frac{\frac{\sqrt{(m-1)}\sqrt{(n+1)}}{\sqrt{(m-1+n+1)}}}{\frac{\sqrt{m}\sqrt{n}}{\sqrt{m+n}}} \qquad \left[\because \beta(m,n) = \frac{\sqrt{m}\cdot\sqrt{n}}{\sqrt{m+n}}\right]$$

$$= \frac{\sqrt{m-1}\cdot\sqrt{n+1}\,\sqrt{m+n}}{\sqrt{m+n}\cdot\sqrt{m}\sqrt{n}}$$

$$= \frac{\sqrt{m-1}\cdot n\cancel{\sqrt{n}}}{\sqrt{m-1+1}\,\cancel{\sqrt{n}}}$$

$$= \frac{\sqrt{m-1}\cdot n}{(m-1)\sqrt{m-1}}$$

$$\boxed{\frac{1}{H} = \frac{n}{m-1}}$$

Imp

*** Mean and variance of Beta distribution of first kind :-**

we know that $\mu_r' = E[x^r]$

$$= \int_0^1 x^r f(x)\, dx$$

$$= \int_0^1 x^r \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\, dx$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^{m+r-1}(1-x)^{n-1}\, dx$$

$$= \frac{1}{\beta(m,n)} \beta(m+r,n)$$

$$= \frac{\beta(m+r,n)}{\beta(m,n)}$$

$$= \frac{\dfrac{\Gamma(m+r)\,\Gamma(n)}{\Gamma(m+r+n)}}{\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

$$= \frac{\Gamma(m+r)\,\Gamma(n)\,\Gamma(m+n)}{\Gamma(m+r+n)\,\Gamma(m)\,\Gamma(n)}$$

$$\boxed{\mu_r' = \frac{\Gamma(m+r)\,\Gamma(m+n)}{\Gamma(m+r+n)\,\Gamma(m)}} \quad -(1)$$

if $r=1$

$$\mu_1' = \frac{\Gamma(m+1)\,\Gamma(m+n)}{\Gamma(m+1+n)\,\Gamma(m)}$$

$$= \frac{(m)\,\cancel{\Gamma(m)}\,\cancel{\Gamma(m+n)}}{(m+n)\,\cancel{\Gamma(m+n)}\,\cancel{\Gamma(m)}}$$

$$\boxed{\mu_1' = \frac{m}{m+n}} \Rightarrow \text{Mean}$$

if $r=2$

$$\mu_2' = \frac{\Gamma(m+2)\,\Gamma(m+n)}{\Gamma(m+n+2)\,\Gamma(m)}$$

$$= \frac{\Gamma(m+1+1)\,\Gamma(m+n)}{\Gamma(m+n+1+1)\,\Gamma(m)}$$

$$= \frac{(m+1)\,\Gamma(m+1)\,\Gamma(m+n)}{(m+n+1)\,\Gamma(m+n+1)\,\Gamma(m)}$$

$$\boxed{\mu_2' = \frac{(m+1)\cdot m\,\Gamma(m)\,\Gamma(m+n)}{(m+n+1)(m+n)\,\Gamma(m+n)\,\Gamma(m)}}$$

$$Var = \mu_2' - (\mu_1')^2$$

$$= \frac{m(m+1)}{(m+n)(m+n+1)} - \left(\frac{m}{m+n}\right)^2$$

$$= \frac{m(m+1)(m+n)^2 - m^2(m+n)(m+n+1)}{(m+n)(m+n)^2(m+n+1)}$$

$$= \frac{\cancel{(m+n)}\left[m\left[(m+1)(m+n) - m(m+n+1)\right]\right]}{\cancel{m+n}\,(m+n)^2(m+n+1)}$$

$$= \frac{m\left[(m+1)(m+n) - m(m+n+1)\right]}{(m+n)^2(m+n+1)}$$

$$= \frac{m\left[\cancel{m^2}+\cancel{mn}+\cancel{m}+n-\cancel{m^2}-\cancel{mn}-\cancel{m}\right]}{(m+n)^2(m+n+1)}$$

$$\boxed{var = \frac{mn}{(m+n)^2(m+n+1)}}$$

* **Mean and variance of Beta distribution of first Second kind :-**

we know that

$$\mu_r' = E(x^r)$$

$$= \int_0^\infty x^r f(x)\, dx$$

$$= \int_0^\infty x^r \frac{1}{B(m,n)} \cdot \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$= \frac{1}{B(m,n)} \int_0^\infty \frac{x^{m+r-1}}{(1+x)^{m+n}}\, dx$$

$$\mu_r' = \frac{1}{B(m,n)} \int_0^\infty \frac{x^{m+r-1}}{(1+x)^{m+n+r-r}}\, dx$$

$$= \frac{1}{B(m,n)} \int_0^\infty \frac{x^{m+r-1}}{(1+x)^{m+r+n-r}}\, dx$$

$$\mu_r' = \frac{\beta(m+r, n-r)}{B(m,n)}$$

$$\mu_r' = \frac{\Gamma(m+r)\,\Gamma(n-r)}{\Gamma(m+r+n-r)} \Big/ \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

$$= \frac{\Gamma(m+r)\,\Gamma(n-r)\,\cancel{\Gamma(m+n)}}{\cancel{\Gamma(m+n)}\,\Gamma m\,\Gamma n}$$

$$\boxed{\mu_r' = \frac{\Gamma(m+r)\,\Gamma(n-r)}{\Gamma m\,\Gamma n}}$$ //

Now $r=1$

$$\mu_1' = \frac{\Gamma(m+1)\,\Gamma(n-1)}{\Gamma m\,\Gamma n}$$

$$= \frac{m\,\cancel{\Gamma m}\,.\,\Gamma(n-1)}{\cancel{\Gamma m}\,\Gamma(n-1+1)}$$

$$= \frac{m\,\cancel{\Gamma(n-1)}}{(n-1)\,\cancel{\Gamma(n-1)}}$$

$$\boxed{\mu_1' = \frac{m}{n-1}}$$ //

put $r=2$

$$\mu_2' = \frac{\Gamma(m+1+1)\,\Gamma(n+2)}{\Gamma m\,\Gamma n}$$

$$= \frac{m+1\,\Gamma(m+1)\,\Gamma(n-2)}{\Gamma m\,\Gamma(n+2-2)}$$

$$= \frac{(m+1)\,m\,\Gamma m\,\cancel{\Gamma(n-2)}}{\Gamma m\,(n-1)\,\Gamma(n-1+(-1))}$$

$$= \frac{(m+1)\, m\, \cancel{\Gamma(n-2)}}{(n-1)(n-2)\cancel{\Gamma(n-2)}}$$

$$\boxed{\mu_2' = \frac{m(m+1)}{(n-1)(n-2)}}$$ //

$$Var = \mu_2' - (\mu_1')^2$$

$$= \frac{m(m+1)}{(n-1)(n-2)} - \left(\frac{m}{n-1}\right)^2$$

$$= \frac{m(m+1)}{(n-1)(n-2)} - \frac{m^2}{(n-1)^2}$$

$$= \frac{m(m+1)(n-1)^2 - m^2(n-1)(n-2)}{(n-1)(n-2)(n-1)^2}$$

$$= \frac{m\cancel{(n-1)}[m(m+1)(n-1) - m^2(n-2)]}{\cancel{(n-1)}(n-2)(n-1)^2}$$

$$= \frac{m[(m+1)(n-1) - m(n-2)]}{(n-1)^2(n-2)}$$

$$= \frac{m[\cancel{mn} - m + n - 1 - \cancel{mn} + 2m]}{(n-1)^2(n-2)}$$

$$\boxed{Var = \frac{m[m+n-1]}{(n-1)^2(n-2)}}$$ //

* Harmonic mean of Beta distribution of first kind:-

we know that

$$\frac{1}{H} = \int_0^1 \frac{1}{x} f(x)\, dx$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^{m-1-1} (1-x)^{n-1} dx$$

$$\frac{1}{H} = \frac{\beta(m-1,n)}{\beta(m,n)}$$

$$\frac{\beta(m-1,n)}{\beta(m,n)} = \frac{\Gamma m-1\ \Gamma n}{\Gamma m+n-1} \cdot \frac{\Gamma m+n}{\Gamma m\ \Gamma n}$$

$$= \frac{\Gamma m-1\ (m+n-1)\ \Gamma m+n-1}{\Gamma m+n-1\ (m-1)\ \Gamma m-1}$$

$$\frac{1}{H} = \frac{m+n-1}{m-1}$$

$$\boxed{H = \frac{m-1}{m+n-1}}$$

29/08/2025 Friday

* limiting case form of gamma distribution:-

Gamma distribution tends to normal distribution as $\lambda \to \infty$

proof: If $x \sim \gamma(\lambda)$, then M.g.f of x is

$$M_X(t) = (1-t)^{-\lambda}$$

consider, standard gamma variate

$$Z = \frac{x-\lambda}{\sqrt{\lambda}}$$

Now consider, M.g.f of standard gamma variate,

$$M_Z(t) = E(e^{tZ})$$

$$= \left[E\, e^{t\left(\frac{x-\lambda}{\sqrt{\lambda}}\right)}\right]$$

$$= E\left[e^{\frac{tx}{\sqrt{\lambda}}} \cdot e^{\frac{-t\lambda}{\sqrt{\lambda}}}\right]$$

$$= e^{\frac{-t\lambda}{\sqrt{\lambda}}} \cdot E\left[e^{\frac{tx}{\sqrt{\lambda}}}\right]$$

$$= e^{\frac{-t\lambda}{\sqrt{\lambda}}} M_X(t/\sqrt{\lambda})$$

$$M_Z(t) = e^{-t\sqrt{\lambda}} \left(1-\frac{t}{\sqrt{\lambda}}\right)^{-\lambda}$$

consider logarithm

$$\log M_Z(t) = -t\sqrt{\lambda} - \lambda \log\left(1-\frac{t}{\sqrt{\lambda}}\right)$$

$$= -t\sqrt{\lambda} + \lambda\left[\frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{3\lambda^{3/2}} + \cdots\right]$$

$$= \frac{t^2}{2} + O(\lambda^{-1/2})$$

$$\lim_{\lambda \to \infty} \log$$

where $o(\lambda^{-1/2})$ is the term containing the powers of λ are $1/2$ and more in the denominator.

Apply limits as $\lambda \to \infty$

$$\lim_{\lambda \to \infty} \log M_Z(t) = \frac{t^2}{2}$$

Consider Exponential,

$$\lim_{\lambda \to \infty} e^{\log M_Z(t)} = e^{t^2/2}$$

$$\lim_{\lambda \to \infty} M_Z(t) = e^{t^2/2}$$

This is M.G.F of standard normal distribution

Hence by uniqueness theorem of M.G.F, Gamma distribution tends to normal distribution as $\lambda \to \infty$

B.M. Shams